10/4

9. Ring of Quaternions: Let us consider the set H of 2×2 complex matrices given by —

$$H = \left\{ \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

$\begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix}$ can be expressed as $aI + bJ + cK + dL$,

where, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $L = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$(H, +, \cdot)$ is a ring with respect to matrix addition and matrix multiplication. This is a non-commutative ring with unity, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ being the unity.

This ring is called the ring of real quaternions.

10. Ring of continuous functions:

Let S be the set of all real valued continuous functions on the closed and bounded interval $[a,b]$.

Let, $f : [a,b] \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ be the elements of S.

We defined addition and multiplication of f and g by

$(f+g)(x) = f(x) + g(x)$, $x \in [a,b]$

$(f.g)(x) = f(x) \cdot g(x)$, $x \in [a,b]$

Then $(S, +, \cdot)$ is a commutative ring with unity. The function i defined by $i(x) = 1$, $\forall x \in [a,b]$ is the unity in the ring. The function o defined by $o(x) = 0$, $\forall x \in [a,b]$, is the zero element in the ring. This ring is denoted by $C[a,b]$.

- Divisor of zero :

Definition : In a ring R, a non-zero element a is said to be a divisor of zero if there exists a non-zero elements b in R such that $a \cdot b = 0$ or a non-zero element c in R such that $c \cdot a = 0$. In the first case, a is said to be a left divisor of zero and in the second case, a is said to be a right divisor of zero.

If, however, R is a commutative ring, every left divisor of zero is also a right divisor of zero and conversely. Thus, there is no distinction between left and right divisor of zero in a commutative ring.

Ex: 1. The ring $(\mathbb{Z}, +, \cdot)$ contains no divisor of zero.

2. The ring $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$ contains no divisor of zero.

3. In the ring $(\mathbb{Z}_6, +, \cdot)$, $\bar{2}, \bar{3}, \bar{4}$ are divisors of zero.

4. The ring $(\mathbb{Z}_5, +, \cdot)$ contains no divisor of zero.

1. Show that the ring of matrices $\left\{ \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ contains divisor of zero and does not contain the unity.

Ans: Let S be the ring and $E = \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}$ in S be the unity.

Then $AE = EA = A$, $\forall$ S in S.

Let $A = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}$. Then $AE = A \Rightarrow \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix} \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 4ax & 0 \\ 0 & 4by \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}$$

$$\Rightarrow 4ax = 2a, \quad 4by = 2b$$

$$\Rightarrow x = 1/2, \quad y = 1/2$$

$$\therefore E = \begin{pmatrix} 2 \cdot 1/2 & 0 \\ 0 & 2 \cdot 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin S$$

$\therefore$ S does not contain the unity.

Let, $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$ are non-zero elements in S. and $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

This shows that S contains divisor of zero.